Visualizing Fuchsian Groups

David Dumas

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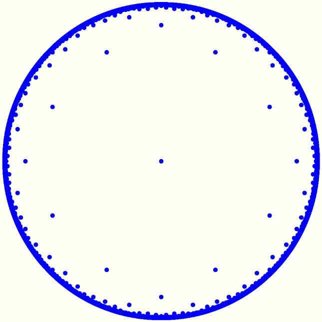
A Fuchsian group is a discrete subgroup $\Gamma < \mathrm{PSL}(2, \mathbb{R})$.

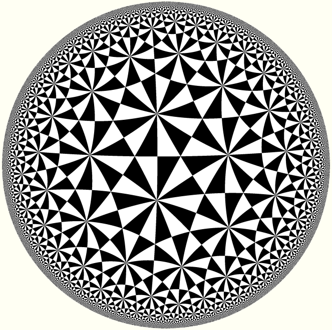
Fuchsian groups correspond to hyperbolic 2-orbifolds, because $\mathrm{PSL}(2, \mathbb{R}) \simeq \mathrm{Isom}^+(\mathbb{H}^2)$.

Question: How can we see a Fuchsian group?

Note: Asking about the set Γ and not the quotient $\mathbb{H}^2/\Gamma$.

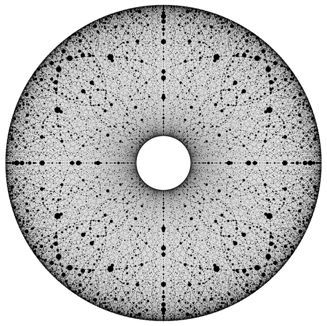
A common approach

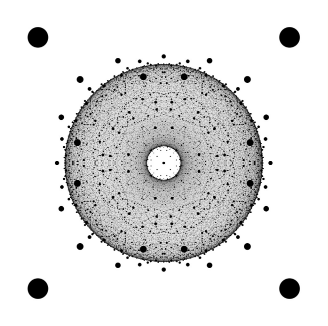




Draw orbits (of points, polygons, etc.) in $\mathbb{H}^2$

Another idea



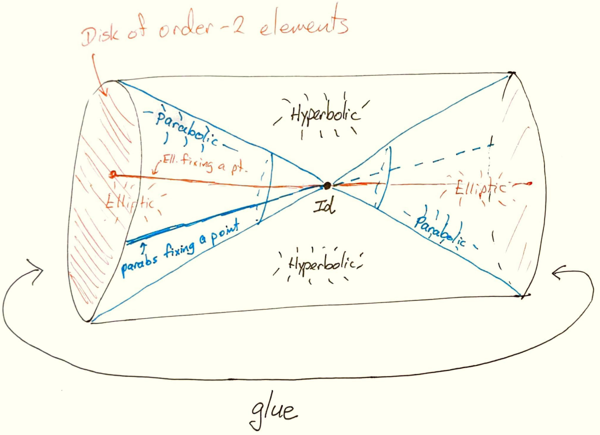


Draw the entire group!

* Of course we will actually draw a large but finite subset.

Cartan: A connected Lie group is diffeomorphic to $K \times \mathbb{R}^n$ where K is a maximal compact subgroup.

E.g. $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathbb{S}^1 \times \mathbb{R}^2 \simeq \mathbb{S}^1 \times \mathbb{D}^2$, an open solid torus.

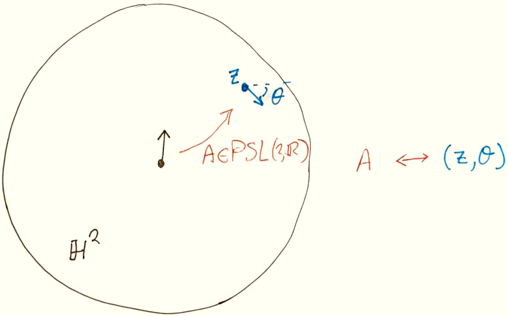


Within this, Γ is a discrete subset.

Hyperbolic interpretation

$\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{T}^1\mathbb{H}^2$ = the unit tangent bundle

The diffeomorphism is the orbit map of a point.



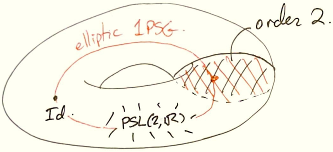
E.g. $A \in \mathrm{PSL}(2,\mathbb{R})$ can be identified with a point in the unit disk model of $\mathbb{H}^2$, and $\theta \in \mathrm{S}^1$ the direction of a tangent vector.

Clifford Torus

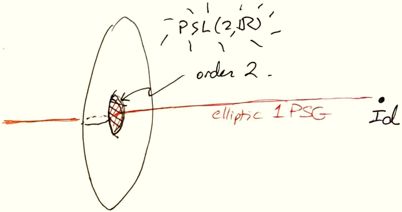
The Clifford torus in $\mathbb{S}^3$ is the set of $v/\|v\|$ where

$$v = (\cos(s), \sin(s), \cos(t), \sin(t)).$$

It divides $\mathbb{S}^3 \simeq \mathbb{R}^3 \cup \{\infty\}$ into two congruent solid tori, either of which can be taken as a model of $\mathrm{PSL}(2, \mathbb{R})$.



Interior torus model



Exterior torus model

In the *exterior torus model*, it is natural to make the point at infinity correspond to the identity element of $\mathrm{PSL}(2, \mathbb{R})$.

Projection Formula

Let $\vec{u} = \left.\frac{\partial}{\partial y}\right|_0$ in the unit disk model of $\mathbb{H}^2$.

For $A \in PSL(2, \mathbb{R})$ let

$$v = \left(\operatorname{Re} A(0), \operatorname{Im} A(0), \frac{\operatorname{Re} A(\vec{u})}{|A(\vec{u})|}, \frac{\operatorname{Im} A(\vec{u})}{|A(\vec{u})|} \right) \in \mathbb{R}^4.$$

E.g. $\quad A = \mathrm{Id} \quad \rightsquigarrow \quad v = (0, 0, 0, 1)$

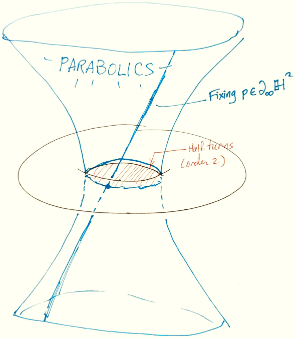
Then apply any stereographic projection to $v/\|v\|$ to get a point $f(A) \in \mathbb{R}^3$.

E.g. Projection $(x, y, z, t) \mapsto \frac{1}{t-1}(x, y, z)$ for an exterior torus model.

Nice Properties

The half-turns (elements of order 2) form a meridian disk.

The parabolic elements give a hyperboloid of one sheet tangent to the boundary torus along a meridian.



Parabolics fixing p $\in \partial_\infty \mathbb{H}^2$ give a straight line.

Implementation

- Main visualizer: Two implementations
  - dumas.io/slview — Three.js particle system
  - PySLView — Python/Cairo for larger offline render jobs

- Utilities etc.
  - `fuchs.py` — Numerically generate a Fuchsian group
  - Triangle group computations — Mathematica, Python
  - Quaternion algebra computations — Magma, Python

Three.js Particle System